MAT 225
Chapter 2-3 Exam Solutions

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Part I
No Calculator

1 Given the matrices below, compute the matrix operations (1) $A + 2B$ and (2) BC, if defined. If an expression is not defined then explain why.

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

$$\begin{aligned} 2B = & \quad 2\begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} & = \begin{bmatrix} 14 & -10 & 2 \\ 2 & -8 & -6 \end{bmatrix} \\ A + 2B = & \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} + \begin{bmatrix} 14 & -10 & 2 \\ 2 & -8 & -6 \end{bmatrix} & = \begin{bmatrix} 16 & -10 & 1 \\ 6 & -13 & -4 \end{bmatrix} \end{aligned} \tag{1.1}$$

(2) BC is undefined because the matrix multiplication BC is only defined where the number of columns in B (which is three) is equal to the number of rows in C (which is two). A mnemonic for this rule is illustrated in (1.2): the bracketed dimensions must be equal.

$$\dim(B)\dim(C) = (2 \times [3)(2] \times 2) \tag{1.2}$$

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2 Compute the product AB.

$$A = \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix}, \; B = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} (3 \times -1) + (-2 \times 2) & (-2 \times -1) + (2 \times 1) \\ (3 \times 5) + (-2 \times 4) & (-2 \times 5) + (1 \times 4) \\ (3 \times 2) + (2 \times 3) & (-2 \times 2) + (1 \times -3) \end{bmatrix}$$

$$= \begin{bmatrix} -3-4 & 2+2 \\ 15-8 & -10+4 \\ 6+6 & -4-3 \end{bmatrix} = \begin{bmatrix} -7 & 4 \\ 7 & -6 \\ 12 & -7 \end{bmatrix}$$

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3 Find the inverse of the matrix A.

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

Augment A with the identity matrix I_3 and take the reduced row echelon form of the resultant augmented matrix. As A is reduced to the identity (assuming it is invertible, this will be possible), I_3 is manipulated by the incident row operations into the inverse of A. The inverse of A, called A^{-1}, can be defined as the composition E of elementary row matrices such that $EA = A^{-1}A = I_3$. Thus row-reducing the augmented matrix $\begin{bmatrix} A & I_3 \end{bmatrix}$ to echelon form produces $\begin{bmatrix} EA & EI_3 \end{bmatrix} = \begin{bmatrix} A^{-1}A & A^{-1}I_3 \end{bmatrix} = \begin{bmatrix} I_3 & A^{-1} \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] 3r_1 + r_2 \to r_2 \quad = \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{-2} & \mathbf{3} & \mathbf{1} & \mathbf{0} \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] (-2)r_1 + r_3 \to r_3 \quad = \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ \mathbf{0} & \mathbf{-3} & \mathbf{8} & \mathbf{-2} & \mathbf{0} & \mathbf{1} \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right] 3r_2 + r_3 \to r_3 \quad = \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & \mathbf{7} & \mathbf{3} & \mathbf{1} \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right] \frac{1}{2}r_3 \to r_3 \quad = \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 1 & \mathbf{7/2} & \mathbf{3/2} & \mathbf{1/2} \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right] 2r_3 + r_2 \to r_2 \quad = \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & \mathbf{10} & \mathbf{4} & \mathbf{1} \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right] 2r_3 + r_1 \to r_1 \quad = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \mathbf{8} & \mathbf{3} & \mathbf{1} \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}$$

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4 Compute the determinant of A by cofactor expansion.

$$A = \begin{bmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 3 & 0 & 0 & 0 \\ 8 & 3 & 1 & 7 \end{bmatrix}$$

Given that A is a 4×4 this could be an arduous calculation involving many steps; luckily A can be manipulated by a single row operation[1] to yield a much easier matrix for cofactor expansion:

$$\begin{bmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 3 & 0 & 0 & 0 \\ 8 & 3 & 1 & 7 \end{bmatrix} (-2)r_4 + r_2 \rightarrow r_2 = \begin{bmatrix} 4 & 0 & \mathbf{0} & 5 \\ -15 & 1 & \mathbf{0} & -19 \\ 3 & 0 & \mathbf{0} & 0 \\ 8 & 3 & \mathbf{1} & 7 \end{bmatrix}$$

Followed by cofactor expansion along the third column, yielding a single 3×3 minor matrix that again contains a convenient $\begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix}$ column to yield the 2×2 minor in one more step:

$$\begin{vmatrix} 4 & 0 & \mathbf{0} & 5 \\ -15 & 1 & \mathbf{0} & -19 \\ 3 & 0 & \mathbf{0} & 0 \\ \not{8} & \not{3} & \mathbf{1} & \not{7} \\ & & \uparrow & \end{vmatrix} \quad \times \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \\ & & \uparrow & \end{bmatrix}$$

$$(-1)\times \begin{vmatrix} 4 & \mathbf{0} & 5 \\ \not{-15} & \mathbf{1} & \not{-19} \\ 3 & \mathbf{0} & 0 \\ & \uparrow & \end{vmatrix} \quad \times \begin{bmatrix} + & - & + \\ - & + & - \\ & \uparrow & \\ + & - & + \end{bmatrix}$$

$$(-1) \times 1 \times \begin{vmatrix} 4 & 5 \\ 3 & 0 \end{vmatrix} \qquad = (4 \times 0) - (5 \times 3) = (-15)$$

$$(-1) \times 1 \times (-15) \qquad = 15$$

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[1] Note that the row replacement operation has no effect on the value of the determinant.

Part II
With Calculator

1 If a matrix A is 5×3 and the product AB is 5×7, what is the size of B?

As seen in 1.2 on page 1, we can answer this by writing out the matrix dimensions as multiplied and considering that (a) the innermost dimensions must match, and (b) the outermost dimensions predict the shape of the product:

$$\begin{aligned} \dim(A)\dim(B) &= (5 \times \mathbf{3})(\boldsymbol{m} \times n) \\ &= (5 \times \mathbf{3})(\mathbf{3} \times n) \\ \dim(AB) &= (5 \times \mathbf{7}) = (5 \times \boldsymbol{n}) \\ \dim(B) &= (m \times n) = (\mathbf{3} \times \mathbf{7}) \end{aligned}$$

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2 ***True*** or ***False***: if A and B are $n \times n$ and invertible, then $A^{-1}B^{-1}$ is the inverse of AB.

False; the inverse of the product AB is given by $(AB)^{-1} = B^{-1}A^{-1}$, paying close attention to the order of multiplication.

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3 Without performing any calculations on the matrix and without using the reduced echelon form, the determinant, or a calculator, determine if the matrix A is invertible and justify that determination.

$$A = \begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix}$$

Even having at hand the tools restricted by the question, we can make a straightforward determination of invertibility by noticing linear dependence between columns of the matrix. We attempt to identify a ratio relating the columns by dividing entries across each row:

$$\begin{bmatrix} -4 \\ 6 \end{bmatrix} \underset{\text{element-wise}}{\div} \begin{bmatrix} 6 \\ -9 \end{bmatrix} = \begin{bmatrix} \frac{-4}{6} \\ \frac{6}{-9} \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$$

therefore

$$\begin{bmatrix} -4 \\ 6 \end{bmatrix} = -\frac{2}{3}\begin{bmatrix} 6 \\ -9 \end{bmatrix}$$

so the columns of A are not linearly independent and A cannot be invertible.

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4 If C is a 6×6 and the equation $C\vec{x} = \vec{v}$ is consistent for every $\vec{v}$ in $\mathbb{R}^6$, is it possible that for some $\vec{v}$, the equation $C\vec{x} = \vec{v}$ has more than one solution? Why or why not?

If $C\vec{x} = \vec{v}$ has a solution $\vec{x}$ for every $\vec{v} \in \mathbb{R}^6$ then $T(\vec{x}) = C\vec{x}$ is a linear transformation $T : \mathbb{R}^6 \to \mathbb{R}^6$ mapping $\mathbb{R}^6$ *onto* $\mathbb{R}^6$ (T is a *surjection*). As the standard matrix of a surjection, C necessarily has linearly independent columns, as each of its 6 columns must be a basis vector in $\mathbb{R}^6$; and so C is an invertible matrix, by properties of invertible matrices. Also by properties of invertible matrices, every invertible matrix has exactly one unique solution for $\vec{x}$ in $C\vec{x} = \vec{v}$; so it is **not possible** that $C\vec{x} = \vec{v}$ has more than one solution for $\vec{x}$, given any $\vec{v}$.

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5 Compute the determinant of A.

$$A = \begin{bmatrix} 1 & 5 & -4 \\ -1 & -4 & 5 \\ -2 & -8 & 7 \end{bmatrix}$$

This is best done on a calculator at this point; on the TI-84,

$$\det([A]) = -3$$

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6 *True* or *False*: the determinant of A is the product of the diagonal entries in A.

False; the determinant of a 2×2 is the difference of downward and upward diagonal products, but this applies only to 2×2 matrices. For matrices larger than 2×2, successive *cofactor expansion* subdivides the matrix into increasingly small submatrices until yielding a set of 2×2 minors whose determinants must be combined according to the method.

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7 Let A and B be 3×3 matrices, with $\det(A) = -3$ and $\det(B) = 4$. Use properties of determinants to calculate $\det(AB)$, $\det(5A)$, and $\det(A^{-1})$.

$$|AB| = |A| \times |B| = -3 \times 4 = -12$$

$$|5A| = 5^3 \times |A| = 125 \times -3 = -375$$
$$|A^{-1}| = \frac{1}{|A|} = -\frac{1}{3}$$

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8 $T(x_1, x_2) = (-5x_1 + 9x_2, 4x_1 - 7x_2)$ is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$. Show that T is invertible and find a formula for T^{-1}.

By properties of invertible matrices, T is necessarily invertible if $T(\vec{x}) = A\vec{x} = \vec{0}$ has exactly one solution for $\vec{x}$ and it is $\vec{x} = \vec{0}$; show that T is invertible by finding that solution and demonstrating that it is the only solution.

First find the standard matrix A of T from the specification of T as a system of linear equations:

$$T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} -5x_1 + 9x_2 \\ 4x_1 - 7x_2 \end{bmatrix} = x_1 \begin{bmatrix} -5 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 9 \\ -7 \end{bmatrix}$$
$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} -5 & 9 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solve $T(\vec{x}) = A\vec{x} = \vec{0}$ by augmenting A and finding the reduced echelon form:

$$\begin{bmatrix} A & \vec{0} \end{bmatrix} = \left[\begin{array}{cc|c} -5 & 9 & 0 \\ 4 & -7 & 0 \end{array}\right] \overset{\text{RREF}}{\longrightarrow} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

There are no free variables in the reduced augmented matrix because all variables have pivot positions in their columns; additionally the value in the third column is the zero vector, so we have shown that $\vec{x} = \vec{0}$ is a solution to $T(\vec{x}) = \vec{0}$ and there is no other solution, so T is an *injective* linear transformation. We have also shown that the columns of A are linearly independent, and that A has 2 pivot positions; by properties of invertible matrices any of these demonstrated traits is sufficient to say that A is an invertible matrix, and so T can be called an invertible transformation.

We find the inverse of T (the inverse of its standard matrix A) as described in question 3 on page 2, by augmenting the standard matrix with the identity and again finding the reduced echelon form:

$$\begin{bmatrix} A & I_2 \end{bmatrix} = \left[\begin{array}{cc|cc} -5 & 9 & 1 & 0 \\ 4 & -7 & 0 & 1 \end{array}\right] \overset{\text{RREF}}{\longrightarrow} \left[\begin{array}{cc|cc} 1 & 0 & 7 & 9 \\ 0 & 1 & 4 & 5 \end{array}\right]$$
$$T^{-1}(\vec{x}) = A^{-1}\vec{x} = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix} \vec{x}$$

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9 Find the area of the parallelogram whose vertices are $(0,0),(-2,4),(4,-5),(2,-1)$.

The area of a parallelogram is equal to the determinant of two of its corners represented by column vectors and adjoined in a matrix:

$$\vec{AB} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \vec{AD} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$A = \begin{vmatrix} \vec{AB} & \vec{AD} \end{vmatrix} = \begin{vmatrix} -2 & 2 \\ 4 & 1 \end{vmatrix} = 6$$

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10 Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1,0,-3)$,$(1,2,4)$, and $(5,1,0)$.

Similar to the area of a parallelogram, the volume of a parallelepiped is equal to the determinant of the matrix formed by adjoining its corner coordinates represented by column vectors:

$$\vec{OA} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \vec{OB} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \vec{OC} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

$$V = \begin{vmatrix} \vec{OA} & \vec{OB} & \vec{OC} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 5 \\ 0 & 2 & 1 \\ -3 & 4 & 0 \end{vmatrix} = 23$$

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11 Determine the values of the parameter s for which the system has a unique solution, and describe the solution.

Given the system of $\vec{x}$ with undetermined coefficients s,

$$\begin{aligned} 6sx_1 + 4x_2 &= 5 \\ 9x_1 + 2sx_2 &= -2 \end{aligned}$$

We solve for $\vec{x}$ in terms of s using *Cramer's Rule.* The requisite matrices of coefficients are

$$A = \begin{bmatrix} 6s & 4 \\ 9 & 2s \end{bmatrix}, A_1 = \begin{bmatrix} 5 & 4 \\ -2 & 2s \end{bmatrix}, A_2 = \begin{bmatrix} 6s & s \\ 9 & -2 \end{bmatrix}$$

and their determinants are

$$\begin{aligned} |A| &= 12s^2 - 36 \\ |A_1| &= 10s + 8 \\ |A_2| &= -12s - 4s \end{aligned}$$

so by Cramer's Rule, the system variables x_1,x_2 are

$$\begin{aligned} x_1 &= \frac{|A_1|}{|A|} = \frac{10s+8}{12(s^2-3)} \\ x_2 &= \frac{|A_2|}{|A|} = \frac{-12s-45}{12(s^2-3)} \end{aligned}$$

and so the solutions $\vec{x}$ to the system are

$$\vec{x} = \begin{bmatrix} \frac{10s+8}{12(s^2-3)} \\ \\ \frac{-12s-45}{12(s^2-3)} \end{bmatrix}, s \neq \pm\sqrt{3}$$