MAT 225
Chapter 4 Exam Solutions

Matt Barnard

1 Determine whether $\vec{w}$ is in the column space of $\boldsymbol{A}$, the null space of $\boldsymbol{A}$, or both.

$$\vec{w} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -3 \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} 7 & 6 & -4 & 1 \\ -5 & -1 & 0 & -2 \\ 9 & -11 & 7 & -3 \\ 19 & -9 & 7 & 1 \end{bmatrix}$$

To test if $\vec{w}$ is in Col $\boldsymbol{A}$ we solve the equation $\boldsymbol{A}\vec{x} = \vec{w}$ for coefficients $\vec{x}$ on the column vectors of $\boldsymbol{A}$; if a solution exists then $\vec{w}$ can be written as a linear combination of the vectors in Col $\boldsymbol{A}$ with weights $\vec{x}$. Augment $\boldsymbol{A}$ with $\vec{w}$ and compute the reduced echelon form of the augmented matrix:

$$[\boldsymbol{A} \quad \vec{w}] = \left[\begin{array}{cccc|c} 7 & 6 & -4 & 1 & 1 \\ -5 & -1 & 0 & -2 & 1 \\ 9 & -11 & 7 & -3 & -1 \\ 19 & -9 & 7 & 1 & -3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & -1/95 & 1/95 \\ 0 & \boxed{1} & 0 & 39/19 & -20/19 \\ 0 & 0 & \boxed{1} & 267/95 & -172/95 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Three of the columns have nonzero pivots (shown in boxes); the fourth column has zero in the pivot position, so the column is linearly dependent on the other three. There are infinitely many solutions for $\vec{x}$ such that $\boldsymbol{A}\vec{x} = \vec{w}$ so $\vec{w}$ is in Col $\boldsymbol{A}$.

The null space of $\boldsymbol{A}$ is defined as the set of vectors $\vec{x}$ that satisfy the equation $\boldsymbol{A}\vec{x} = \vec{0}$; $\vec{w}$ is therefore in Nul $\boldsymbol{A}$ if $\boldsymbol{A}\vec{w} = \vec{0}$. Multiplying, we have

$$\boldsymbol{A}\vec{w} = \begin{bmatrix} 7 & 6 & -4 & 1 \\ -5 & -1 & 0 & -2 \\ 9 & -11 & 7 & -3 \\ 19 & -9 & 7 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

so $\vec{w}$ is not in Nul $\boldsymbol{A}$.

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2 Find a basis for the space spanned by the vectors below and determine the dimension of the space.

$$\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -6 \\ 9 \end{bmatrix}$$

(1) The vectors forming a basis are necessarily linearly independent, by the definition of a basis (Lay p. 211).

(2) The *spanning set theorem* ($\boxed{5a}$ in Lay p. 212) states that if any vector $\vec{v}$ in a set S is a linear combination of some others vectors in S, then the span of $S \setminus \vec{v}$ is the same as the span of S.

We satisfy (1) by choosing all of the vectors above that are linearly independent, and (2) ensures that those choices form a basis for the space V spanned by all of the above vectors. We can identify linear independence by writing the vectors as columns of a matrix $\boldsymbol{A}$ and taking the reduced echelon form of $\boldsymbol{A}$; dependent columns have 0 in their pivot positions.

$$\begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -3 & 0 & 4 \\ 0 & 1 & -4 & 0 & -5 \\ 0 & 0 & \boxed{0} & 1 & -2 \\ 0 & 0 & 0 & 0 & \boxed{0} \end{bmatrix}$$

We see that the third and fifth vectors have zeros in their pivot positions (indicated by boxes) so they are linearly dependent in this set. The set of vectors one, two, and four forms a basis $\mathcal{B}$ for V by the spanning set theorem ($\boxed{5b}$ in Lay p. 212):

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix} \right\}$$

The dimension of the space V is defined as the number of vectors in any basis for V (Lay p. 228). The number of vectors in $\mathcal{B}$ is three, so the dimension of V is 3.

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3 Find the vector $\vec{x}$ determined by the given coordinate vector $[\vec{x}]_{\mathcal{B}}$ and the given basis $\mathcal{B}$.

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 0 \end{bmatrix} \right\}, [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}_{\mathcal{B}}$$

With no basis specified on $\vec{x}$ we can assume it to be in standard basis ($\mathcal{E}$) coordinates. $[\vec{x}]_\mathcal{B}$ is in $\mathcal{B}$-coordinates so we transform $[\vec{x}]_\mathcal{B}$ to $\vec{x}$ using the change-of-coordinates matrix $\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}}$, which we construct by writing the $\mathcal{B}$ basis vectors $\left\{\vec{b}_1, \vec{b}_2, \vec{b}_3\right\}$, transformed to $\mathcal{E}$ coordinates, as matrix columns. The order of the columns in $\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}}$ corresponds to the order of coordinates in $[\vec{x}]_\mathcal{B}$. The $\mathcal{B}$ vectors are already given in $\mathcal{E}$ coordinates, so we simply write

$$\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \vec{b}_3\end{bmatrix} = \begin{bmatrix} 1 & 5 & 4 \\ -4 & 2 & -7 \\ 3 & -2 & 0 \end{bmatrix}$$

$$\vec{x} = \boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}}\,[\vec{x}]_\mathcal{B} = \begin{bmatrix} 1 & 5 & 4 \\ -4 & 2 & -7 \\ 3 & -2 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}_\mathcal{B} = \begin{bmatrix} -1 \\ -5 \\ 9 \end{bmatrix}$$

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4 Find the coordinate vector $[\vec{x}]_\mathcal{B}$ of $\vec{x}$ relative to the given basis $\mathcal{B} = \left\{\vec{b}_1, \vec{b}_2, \vec{b}_3\right\}$.

$$\vec{b}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix}, \vec{b}_3 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \vec{x} = \begin{bmatrix} 8 \\ -9 \\ 6 \end{bmatrix}$$

The process here is the same as in Question 3, but instead of taking vectors from $\mathcal{B}$ to $\mathcal{E}$ coordinates we need a transform from $\mathcal{E}$ to $\mathcal{B}$. This is the inverse of the type of transformation used in Question 3 and is performed by the inverse of such a change-of-coordinate matrix:

$$\boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{E}} = \boldsymbol{P}^{-1}_{\mathcal{E}\leftarrow\mathcal{B}}$$

Now the basis here is different from the one in Question 3, so we first construct $\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}}$ so we can take its inverse:

$$\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & -3 & 2 \\ -1 & 4 & -2 \\ -3 & 9 & 4 \end{bmatrix}$$

$$\boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{E}} = \boldsymbol{P}^{-1}_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{bmatrix} 17/5 & 3 & -1/5 \\ 1 & 1 & 0 \\ 3/10 & 0 & 1/10 \end{bmatrix}$$

and $\vec{x}$ is transformed by matrix multiplication:

$$[\vec{x}]_\mathcal{B} = \boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{E}}\vec{x} = \boldsymbol{P}^{-1}_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{bmatrix} 17/5 & 3 & -1/5 \\ 1 & 1 & 0 \\ 3/10 & 0 & 1/10 \end{bmatrix} \begin{bmatrix} 8 \\ -9 \\ 6 \end{bmatrix}$$

$$[\vec{x}]_\mathcal{B} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

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5 Determine if $\{3+7t, 5+t-2t^3, t-2t^2, 1+16t-6t^2+2t^3\}$ form a basis for $\mathbb{P}_3$.

We're given four polynomials of highest degree 3 so they may form a basis for $\mathbb{P}_3$ if they are linearly independent. Looking at the standard basis for $\mathbb{P}_3$, $\{1, x, x^2, x^3\}$, we can see that $\dim \mathbb{P}_3 = 4$, so all of the given polynomials must be independent vectors in $\mathbb{P}_3$ to form a basis for it. We write the polynomials as coordinate vectors in $\mathbb{P}_3$; the coefficient a in at^p is the coordinate on the pth-power vector in the standard basis.

$$\left\{\vec{b}_i\right\} = \left\{ \begin{bmatrix} 3 \\ 7 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 16 \\ -6 \\ 2 \end{bmatrix} \right\}$$

Then we test independence by writing the vectors as columns in a matrix and taking its reduced echelon form:

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \vec{b}_4 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 0 & 1 \\ 7 & 1 & 1 & 16 \\ 0 & 0 & -2 & -6 \\ 0 & 2 & 0 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

Each vector has a pivot in the reduced echelon matrix so they are independent. By the *basis theorem*, any linearly independent set of p vectors in a p-dimensional vector space is a basis for that space ($\boxed{12}$ in Lay p. 229). The vectors $\left\{\vec{b}_1, \ldots, \vec{b}_4\right\}$ are in $\mathbb{P}_3$ and $\dim \mathbb{P}_3 = 4$ so $\left\{\vec{b}_1, \ldots, \vec{b}_4\right\}$ is a basis for $\mathbb{P}_3$.

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6 Assume that the matrix $\boldsymbol{A}$ is row equivalent to $\boldsymbol{B}$. List $\operatorname{rank} \boldsymbol{A}$ and $\dim \operatorname{Nul} \boldsymbol{A}$; then find bases for $\operatorname{Col} \boldsymbol{A}$, $\operatorname{Row} \boldsymbol{A}$, and $\operatorname{Nul} \boldsymbol{A}$.

$$\boldsymbol{A} = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \boxed{1} & -3 & 0 & 5 & -7 \\ 0 & 0 & \boxed{2} & -3 & 8 \\ 0 & 0 & 0 & 0 & \boxed{5} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are row-equivalent then $\boldsymbol{B}$ can be considered an echelon form of $\boldsymbol{A}$. The pivot positions in $\boldsymbol{B}$ are indicated with boxes, and these correspond to linearly independent columns in $\boldsymbol{A}$. We find $\operatorname{rank} \boldsymbol{A}$ easily by counting these, as $\operatorname{rank} \boldsymbol{A} = \dim \operatorname{Col} \boldsymbol{A} = 3$. By the *rank theorem*, the null space of an $m \times n$ $\boldsymbol{M}$

has a dimension such that $\operatorname{rank} \boldsymbol{M} + \dim \operatorname{Nul} \boldsymbol{M} = n$. The dimension of $\operatorname{Nul} \boldsymbol{A}$ is therefore $5 - \operatorname{rank} \boldsymbol{A} = 5 - 3 = 2$.

A basis for $\operatorname{Col} \boldsymbol{A}$ is simply the vectors in $\boldsymbol{A}$ that correspond to the pivot columns in $\boldsymbol{B}$ (row operations do not change the correspondence of these vectors):

$$\text{basis } \operatorname{Col} \boldsymbol{A} = \left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 9 \\ -10 \\ -3 \\ 0 \end{bmatrix} \right\}$$

A basis for $\operatorname{Row} \boldsymbol{A}$ is simply the linearly independent columns of $\boldsymbol{A}^T$. A basis for $\operatorname{Row} \boldsymbol{A}$ cannot, however, be drawn from $\boldsymbol{A}$ because row operations change the correspondence between rows in $\boldsymbol{A}$ and rows in $\boldsymbol{B}$ (e.g. two rows in $\boldsymbol{A}$ may be swapped in $\boldsymbol{B}$). We draw the basis for $\operatorname{Row} \boldsymbol{A}$ then from the columns not of $\boldsymbol{A}^T$, but of $\boldsymbol{B}^T$, using the same pivot positions from $\boldsymbol{B}$:

$$\text{basis } \operatorname{Row} \boldsymbol{A} = \left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \\ 5 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ -3 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} \right\}$$

To find a basis for $\operatorname{Nul} \boldsymbol{A}$ we first express the matrix equation $\boldsymbol{B}\vec{x} = \vec{0}$ in parametric form, letting t and s equal the respective free variables in $\vec{x}$:

$$\begin{aligned} x_1 &= 3t - 5 \\ x_2 &= t \\ x_3 &= {}^3\!/\!_2 s \\ x_4 &= s \\ x_5 &= 0 \end{aligned}$$

$$\operatorname{Nul} \boldsymbol{A} = t \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -5 \\ 0 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}$$

A basis for $\operatorname{Nul} \boldsymbol{A}$ is formed by the vectors above:

$$\text{basis } \operatorname{Nul} \boldsymbol{A} = \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 3/2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

7 Let $\mathcal{B} = \left\{\vec{b}_1, \vec{b}_2\right\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ be bases for a vector space V, and suppose that $\vec{b}_1 = 6\vec{c}_1 - 2\vec{c}_2$ and $\vec{b}_2 = 9\vec{c}_1 - 4\vec{c}_2$. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ and then find $[\vec{x}]_\mathcal{C}$ for $\vec{x} = -3\vec{b}_1 + 2\vec{b}_2$.

We construct the change-of-coordinates matrix as in Questions 3 and 4. The coordinates of the $\mathcal{B}$ vectors are given in terms of $\mathcal{C}$, so we write them outwrite:

$$\boldsymbol{P}_{\mathcal{C}\leftarrow\mathcal{B}} = \left[\left[\vec{b}_1\right]_\mathcal{C} \quad \left[\vec{b}_2\right]_\mathcal{C}\right] = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}$$

$\vec{x}$ is given in terms of the $\mathcal{B}$ vectors, so transform them directly by matrix multiplication:

$$[\vec{x}]_\mathcal{C} = \boldsymbol{P}_{\mathcal{C}\leftarrow\mathcal{B}} [\vec{x}]_\mathcal{B} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

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8 Let $\mathcal{B} = \left\{\begin{bmatrix} 7 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \end{bmatrix}\right\}$ and $\mathcal{C} = \left\{\begin{bmatrix} 1 \\ -5 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}\right\}$ be bases for $\mathbb{R}^2$. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ and the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

$$[\mathcal{B} \quad \mathcal{C}] = \left[\begin{array}{cc|cc} 7 & -3 & 1 & -2 \\ 5 & -1 & -5 & 2 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & -5 & 3 \end{array}\right]$$

$$\boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$$
$$\boldsymbol{P}_{\mathcal{C}\leftarrow\mathcal{B}} = \boldsymbol{P}_{B\leftarrow C}^{-1} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}$$

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9 Answer *True* or *False*:

A) If $H = \text{Span}\left\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_p\right\}$ then $\left\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_p\right\}$ is a basis for H.

False; some subset of $\left\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_p\right\}$ is a basis for H, but not all vectors in the set are necessarily independent.

B) If $\boldsymbol{B}$ is an echelon form of a matrix $\boldsymbol{A}$ then the pivot columns of $\boldsymbol{B}$ form a basis for $\operatorname{Col}\boldsymbol{A}$.

False; the pivot columns in $\boldsymbol{B}$ correspond to the vectors in $\boldsymbol{A}$ that form a basis for $\operatorname{Col}\boldsymbol{A}$.

C) The vector spaces $\mathbb{P}_3$ and $\mathbb{R}^3$ are isomorphic.

False; $\mathbb{P}_3$ is the four-dimensional space of degree 3 polynomials and $\mathbb{R}^3$ is a three-dimensional space.

D) $\mathbb{R}^2$ is a two-dimensional subspace of $\mathbb{R}^3$.

False

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10 Let $\boldsymbol{A}$ be the following matrix; find n such that $\operatorname{Nul}\boldsymbol{A}$ is a subspace of $\mathbb{R}^n$ and find m such that $\operatorname{Col}\boldsymbol{A}$ is a subspace of $\mathbb{R}^m$.

$$\boldsymbol{A} = \begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ 3 & -9 \end{bmatrix}$$

$\operatorname{Nul}\boldsymbol{A}$ is a subspace of $\mathbb{R}^2$ so $n = 2$. $\operatorname{Col}\boldsymbol{A}$ is a subspace of $\mathbb{R}^4$ so $m = 4$.

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11 Determine if the set of all polynomials of degree at most 3, with integers as coefficients, is a subspace of $\mathbb{P}_3$.

This set S is not a subspace of $\mathbb{P}_3$. A subspace of $\mathbb{P}_3$ must be closed under vector addition, closed under scalar multiplication, and contain the zero vector. S is not closed under scalar multiplication because not all scalar multiples of vectors in S will have integer coefficients; specifically $c\vec{p}(t)$ where $\vec{p} \in S$ does not have an integer coefficient when $0 < c < 1$.

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12 A) For a given set V with vector addition and scalar multiplication, write the ten conditions that must be true for V to be called a vector space.

1. V is closed under vector addition
2. Vector addition in V commutes
3. Vector addition in V is associative
4. There is a zero vector $\vec{0}$ in V such that $\vec{0} + \vec{u} = \vec{u}$ where $\vec{u} \in V$
5. For each $\vec{u} \in V$ there is a negation $(-\vec{u})$ such that $\vec{u} + (-\vec{u}) = \vec{0}$.
6. V is closed under scalar multiplication
7. Scalar multiplication distributes over vector addition in V: $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
8. Vector multiplication distributes over scalar addition: $\vec{u}(c + d) = \vec{u}c + \vec{u}d$
9. Scalar multiplication is associative: $c(d\vec{u}) = (cd)\vec{u}$
10. There is a scalar multiplicative identity 1 such that $1 \cdot \vec{u} = \vec{u}$

B) For a given constant ω, let

$$S = \{y(t) = c_1 \cdot \cos(\omega t) + c_2 \cdot \sin(\omega t) \mid c_1, c_2 \in \mathbb{R}\}$$

This set of functions is a vector space. Prove the necessary conditions to show that S is a subspace of the vector space of all real valued functions.

1. S is closed under vector addition

$$\begin{aligned}\vec{p}(t) + \vec{q}(t) &= (p_1 \cos(\omega t) + p_2 \sin(\omega t)) + (q_1 \cos(\omega t) + q_2 \sin(\omega t)) \\ &= (p_1 + q_1)\cos(\omega t) + (p_2 + q_2)\sin(\omega t) \\ &= x_1 \cos(\omega t) + x_2 \sin(\omega t) \\ &= \vec{x}(t) \in S\end{aligned}$$

2. S is closed under scalar multiplication

$$\begin{aligned}c\vec{p}(t) &= c\,(p_1 \cos(\omega t) + p_2 \sin(\omega t)) \\ &= cp_1 \cos(\omega t) + cp_2 \sin(\omega t) \\ &= x_1 \cos(\omega t) + x_2 \sin(\omega t) \\ &= \vec{x}(t) \in S\end{aligned}$$

3. S contains the zero vector

$$c\vec{p}(t) \in S, c = 0 \Rightarrow \vec{0} \in S$$