MAT 231
Extra Credit

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1 Explain how the numbers in the Fibonacci sequence are derived from the growth of a rabbit population

Leonardo of Pisa, now known as Fibonacci, described the growth of a population of rabbits as

> (1) beginning with a single pair, (2) every month each pair produces a new pair; (3) each new pair becomes productive in its second month.

He gave a formula for the size of such a population that produces the sequence of *Fibonacci Numbers*, the *Fibonacci Sequence*, stated as

$$\begin{aligned} f_0 &= 1 \\ f_1 &= 1 \\ f_n &= f_{n-1} + f_{n-2} \end{aligned}$$

where f_n is the nth Fibonacci Number and represents the number of breeding pairs in the population at the nth month. The population at each month comprises two distinct groups: breeding and non-breeding pairs. A breeding pair is defined by part (3) of the above statement; all others are non-breeding.

The first Fibonacci number, f_0, represents an initial condition of the population: it spends its first month with a count of one new non-breeding pair, according to part (1) of the statement. In the second month the population has a count f_1 of one breeding pair plus zero non-breeding pairs. Note that the first month is the zeroth index on f, and that each new pair breeds one time during its second month *after* the count for that month; thus the first offspring appear in f_1, but are not counted until f_2. In the next month there exists the initial pair, having become a breeding pair in the previous month, and its offspring from that month. We count this by $f_n = f_{n-1} + f_{n-2}$ where f_{n-1} is all pairs, breeding and non-breeding, from all previous months; and f_{n-2} is one new pair for each pair who has achieved breeding status.

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2 Find a power series for the Fibonacci numbers and its radius of convergence

The previous formula is recursive in that f_n is determined by $\{f_{n-1}, f_{n-2}, \cdots, f_{n-n}\}$ and so each value in the sequence must be computed for any n. We will develop a *closed form* for this sequence, that being a function that produces f_n for any n without knowing f_n for any other n. Consider the function $F(x)$ defined by the power series below, whose coefficients of terms are the ordered Fibonacci numbers:

$$F(x) = \sum_{n=0}^{\infty} f_n x^n = 1 + 1x + 2x^2 + 3x^3 + 5x^4 + \cdots$$

It can be proven that this series, centered at $x = 0$, converges by the fact that

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2}$$

This limit is a factor in the *ratio test for series convergence* when $a_n = f_n x^n$, which has the general form

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

and states that the series $\sum_{n=0}^{\infty} a_n$ converges if and only if $L < 1$. Applying that test to the above series we have

$$L = \lim_{n\to\infty} \left| \frac{f_{n+1}x^{n+1}}{f_n x^n} \right| = \lim_{n\to\infty} \left| \left(\frac{f_{n+1}}{f_n} \right) x \right| = \left| \left(\frac{1+\sqrt{5}}{2} \right) x \right|$$

Solving $L < 1$ for x gives us

$$\begin{aligned} \left| \left(\frac{1+\sqrt{5}}{2} \right) x \right| &< 1 \\ |x| &< \frac{2}{1+\sqrt{5}} \left(\frac{1-\sqrt{5}}{1-\sqrt{5}} \right) = \frac{2-2\sqrt{5}}{-4} \\ |x| &< \frac{1}{2}(\sqrt{5}-1) \\ R &= \frac{1}{2}(\sqrt{5}-1) \end{aligned}$$

So the series $\sum_{n=0}^{\infty} f_n x^n$ converges to $F(x)$ when $|x| < R$. Whether the series converges at $|x| = R$ cannot be determined in this way because then $L = 1$ and the ratio test is inconclusive.

3 Find a function that satisfies $F(x) = \sum_{n=0}^{\infty} f_n x^n$

It can be shown from the relation $f_n = f_{n-1} + f_{n-2}$ that

$$(x^2 + x - 1)F(x) = -1$$

Writing terms of $F(x)$ and distributing, we have

$$\begin{array}{rrrrrrrrrr} & \multicolumn{9}{c}{\left(x^2+x-1\right)\left(1+1x+2x^2+3x^3+5x^4+\cdots\right)} \\ = & & & 1x^2 & +1x^3 & +2x^4 & +3x^5 & +5x^6 & +\cdots \\ & & +1x & +1x^2 & +2x^3 & +3x^4 & +5x^5 & +8x^6 & +\cdots \\ & -1 & -1x & -2x^2 & -3x^3 & -5x^4 & -8x^5 & -13x^6 & -\cdots \\ = & -1 \end{array}$$

Multiplying the nth term $f_n x^n$ by x^p increases the power of the term to $n+p$ so that the coefficient f_n appears in the term $f_n x^{n+p}$. Thus by multiplying every term by x^p we can advance each coefficient by p positions in the sequence of terms.

Now the definition of f_n implies that any term x^n has as its coefficient the sum of the coefficients of the previous two terms. Multiplying every term by x, we generate a polynomial where the x^n term has coefficient f_{n-1}; likewise multiplying by x^2 generates a polynomial with the term $f_{n-2}x^n$. Adding those two polynomials then yields the original, $F(x)$.

When we multiply $(x^2+x-1)F(x)$ we are then generating three polynomials: one with f_{n-2} on the nth term, one with f_{n-1} on the nth term, and one with $-f_n$ on the nth term. Adding these we have $f_{n-2} + f_{n-1} - f_n$, or $f_n - f_n = 0$; thus each term is eliminated save one: the f_0 coefficient is shifted into the $n = 1$ position by $xF(x)$ and into the $n = 2$ position by $x^2F(x)$. The 0th term in $F(x)$ therefore has no term added to it and so it remains in the final sum as $(-1)f_0x^0 = -1$.

Solving for $F(x)$ gives us an algebraic function for the series in the previous section:

$$F(x) = \frac{-1}{x^2 + x - 1}$$

Neither the previous series nor this function enables us to compute the value of f_n except by iterating n. We can, however, manipulate this function to produce an equivalent one that has a series representation with a known formula for arbitrary coefficients. We begin by factoring:

$$F(x) = \frac{-1}{(x + w_1)(x + w_2)}$$

$$w = \frac{-(1) \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}$$

$$F(x) = \frac{-1}{\left(x - \frac{1+\sqrt{5}}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right)}$$

followed by partial fraction decomposition,

$$F(x) = \frac{A}{x - \frac{1+\sqrt{5}}{2}} + \frac{B}{x - \frac{1-\sqrt{5}}{2}}$$

$$-1 = A\left(x - \frac{1+\sqrt{5}}{2}\right) + B\left(x - \frac{1-\sqrt{5}}{2}\right)$$

$$\text{let } x = \frac{1+\sqrt{5}}{2} \qquad -1 = A(0) + B(\sqrt{5})$$

$$\boxed{-\frac{1}{\sqrt{5}}} = B$$

$$\text{let } x = \frac{1-\sqrt{5}}{2} \qquad -1 = A(-1) + B(0)$$

$$\boxed{\frac{1}{\sqrt{5}}} = A$$

$$F(x) = \frac{5^{-1/2}}{x - \frac{1+\sqrt{5}}{2}} - \frac{5^{-1/2}}{x - \frac{1-\sqrt{5}}{2}}$$

Further manipulation yields two terms in the form of the geometric series, $\frac{1}{1-x} = \sum x^n$:

$$\text{let } w_1 = \frac{1+\sqrt{5}}{2}, \quad w_2 = \frac{1-\sqrt{5}}{2}$$

$$\begin{aligned}
\sqrt{5}F(x) &= \frac{-1}{-1}\left(\frac{1}{x - w_1} - \frac{1}{x - w_2}\right) \\
&= \frac{-1}{w_1 - x} - \frac{-1}{w_2 - x} \\
&= \frac{1}{w_2\left(1 - \frac{x}{w_2}\right)} - \frac{1}{w_1\left(1 - \frac{x}{w_1}\right)} \\
&= \left(w_2^{-1}\frac{1}{1 - \frac{x}{w_2}}\right) - \left(w_1^{-1}\frac{1}{1 - \frac{x}{w_1}}\right)
\end{aligned}$$

Writing as their series representations,

$$\begin{aligned}
\sqrt{5}F(x) &= w_2^{-1}\sum_{n=0}^{\infty}\left(\frac{x}{w_2}\right)^n - w_1^{-1}\sum_{n=0}^{\infty}\left(\frac{x}{w_1}\right)^n \\
&= \sum_{n=0}^{\infty} w_2^{-n-1}x^n - \sum_{n=0}^{\infty} w_1^{-n-1}x^n
\end{aligned}$$

and adding,

$$F(x) = \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \left(w_2^{-n-1} - w_1^{-n-1} \right) x^n$$

$$F(x) = \sum_{n=0}^{\infty} \left(\frac{5^{-1/2}}{\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1+\sqrt{5}}{2}\right)^{n+1}} \right) x^n$$

Recalling the first definition of $F(x)$,

$$F(x) = \sum_{n=0}^{\infty} f_n x^n$$

where the nth coefficient f_n is the nth Fibonacci number; then

$$f_n = \frac{5^{-1/2}}{\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1+\sqrt{5}}{2}\right)^{n+1}}$$

In the new power series representation we have a rather more elaborate formula for the coefficient, but it is no longer recursive—meaning we can find any Fibonacci number in constant—instead of linear—computation time. We can guarantee these series are equivalent because every analytic function has exactly one Taylor series representation.