## 1(a) $\dfrac{dy}{dx} + \dfrac{3}{x}y = e^{2x}$

This is a linear ODE in standard form:

$$\frac{dy}{dx} + P(x)y = f(x) \tag{1}$$

$$P(x) = \frac{3}{x},\ f(x) = e^{2x}$$

### Short version

Use $y \cdot e^{\int P(x)\,\mathrm{d}x} = \int e^{\int P(x)\,\mathrm{d}x} f(x)\,\mathrm{d}x$:

$$\int P(x)\,\mathrm{d}x = 3\int \frac{1}{x}\,\mathrm{d}x = 3\ln x$$

$$e^{\int P(x)\,\mathrm{d}x} = e^{3\ln x} = \left(e^{\ln x}\right)^3 = x^3$$

$$y \cdot x^3 = \int x^3 e^{2x}\,\mathrm{d}x$$

and skip to the integration at the end.

### Long version

Zill emphasized on p. 55 of our book that

> ... you should **not memorize** [the formula above]. The following procedure should be worked through each time.

That procedure is:

Find the integrating factor $\mu(x) = e^{\int P(x)\,\mathrm{d}x}$

$$\int P(x)\,\mathrm{d}x = \int 3\frac{1}{x}\,\mathrm{d}x = 3\ln x$$

$$\begin{aligned}\mu(x) &= e^{\int P(x)\,\mathrm{d}x} = e^{3\ln x} = \left(e^{\ln x}\right)^3 \\ &= x^3 \end{aligned} \tag{2}$$

and multiply both sides of the standard-form DE (1) by the integrating factor

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)f(x) \tag{3}$$

Now $\mu(x) = e^{\int P(x)\,\mathrm{d}x}$ so, by the chain rule,

$$\begin{aligned}\frac{d}{dx}\mu(x) &= e^{\int P(x)\,\mathrm{d}x} \cdot \frac{d}{dx}\int P(x)\,\mathrm{d}x \\ &= e^{\int P(x)\,\mathrm{d}x} P(x) \\ &= \mu(x)P(x)\end{aligned} \tag{4}$$

Substituting (4) into (3) we have

$$\mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y = \mu(x)f(x) \tag{5}$$

which has the derivative of a product on the left hand side

$$\frac{d}{dx}\left[y \cdot \mu(x)\right] = \frac{dy}{dx}\mu(x) + \frac{d\mu}{dx}y \tag{6}$$

Substituting (6) into (5) then gives us

$$\frac{d}{dx}\Big[y \cdot \mu(x)\Big] = \mu(x) f(x)$$

so we can integrate on both sides to find $y$ (hence the name *integrating factor*)

$$\int \frac{d}{dx}\Big[y \cdot \mu(x)\Big]\,\mathrm{d}x = \int \mu(x) f(x)\,\mathrm{d}x$$
$$y \cdot \mu(x) = \int \mu(x) f(x)\,\mathrm{d}x$$

Finally we substitute $\mu(x)$ from (2) and $f(x)$ from (1) and integrate

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$$y \cdot x^3 = \int x^3 e^{2x}\,\mathrm{d}x$$

$$\left\{\begin{array}{ll} u = x^3 & dv = e^{2x}dx \\ du = 3x^2 dx & v = \tfrac{1}{2}e^{2x} \end{array}\right\} = \overset{M}{\boxed{\frac{1}{2}e^{2x}x^3}} - \int \frac{1}{2}e^{2x}3x^2\,\mathrm{d}x$$

$$= M - \frac{3}{2}\int e^{2x}x^2\,\mathrm{d}x$$

$$\left\{\begin{array}{ll} u = x^2 & dv = e^{2x}dx \\ du = 2x dx & v = \tfrac{1}{2}e^{2x} \end{array}\right\} = M - \frac{3}{2}\left(\frac{1}{2}e^{2x}x^2 - \int \frac{1}{2}e^{2x}2x\,\mathrm{d}x\right)$$

$$= M - \overset{N}{\boxed{\frac{3}{4}e^{2x}x^2}} + \frac{3}{2}\int e^{2x}x\,\mathrm{d}x$$

$$\left\{\begin{array}{ll} u = x & dv = e^{2x}dx \\ du = dx & v = \tfrac{1}{2}e^{2x} \end{array}\right\} = M - N + \frac{3}{2}\left(\frac{1}{2}e^{2x}x - \int \frac{1}{2}e^{2x}dx\right)$$

$$= M - N + \frac{3}{4}e^{2x}x - \frac{3}{8}e^{2x} + C$$

$$y \cdot x^3 = M - N + \cdots$$
$$= \frac{1}{2}e^{2x}x^3 - \frac{3}{4}e^{2x}x^2 + \frac{3}{4}e^{2x}x - \frac{3}{8}e^{2x} + C$$
$$= \frac{1}{8}e^{2x}\left(4x^3 - 6x^2 + 6x - 3\right) + C$$
$$\frac{y \cdot x^3}{x^3} = \frac{1}{8}e^{2x}\left(\frac{4x^3}{x^3} - \frac{6x^2}{x^3} + \frac{6x}{x^3} - \frac{3}{x^3}\right) + \frac{C}{x^3}$$

$$\text{(a)}\ \boxed{y = \frac{1}{8}e^{2x}\left(4 - 6x^{-1} + 6x^{-2} - 3x^{-3}\right) + Cx^{-3}}$$

## 1(b) $\dfrac{dy}{dx} = 8x + 2xy$

Rearrange to write in standard linear form

$$\frac{dy}{dx} - 2xy = 8x$$
$$P(x) = -2x,\ f(x) = 8x$$

### Short version

$$y \cdot e^{\int P(x)\,\mathrm{d}x} = \int e^{\int P(x)\,\mathrm{d}x} f(x)\,\mathrm{d}x$$

$$\int P(x)\,\mathrm{d}x = \int -2x\,\mathrm{d}x = -x^2$$
$$e^{\int P(x)\,\mathrm{d}x} = e^{-x^2}$$

$$y \cdot e^{-x^2} = \int e^{-x^2} 8x\,\mathrm{d}x$$

then skip to the last integration step.

### Long version

Find the integrating factor

$$\begin{aligned}\mu(x) &= e^{\int P(x)\,\mathrm{d}x} = e^{\int -2x\,\mathrm{d}x} \\ &= e^{-x^2}\end{aligned}$$

and multiply both sides of the DE

$$\frac{dy}{dx}e^{-x^2} \boxed{-e^{-x^2}2x} y = e^{-x^2}8x \tag{7}$$

The derivative of the integrating factor is

$$\begin{aligned}\frac{d}{dx}\left[e^{-x^2}\right] &= e^{-x^2} \cdot \frac{d}{dx}\left[-x^2\right] \\ &= \boxed{-e^{-x^2}2x}\end{aligned} \tag{8}$$

Substitute (8) into (7)

$$\frac{dy}{dx}e^{-x^2} + \frac{d}{dx}\left[e^{-x^2}\right] y = e^{-x^2}8x \tag{9}$$

The left hand side is the derivative of a product:

$$\frac{dy}{dx}e^{-x^2} + \frac{d}{dx}\left[e^{-x^2}\right] y = \frac{d}{dx}\left[y \cdot e^{-x^2}\right]$$

substitute that product into (9)

$$\frac{d}{dx}\left[y \cdot e^{-x^2}\right] = e^{-x^2}8x$$

and integrate both sides

$$\begin{aligned}\int \frac{d}{dx}\left[y \cdot e^{-x^2}\right]\mathrm{d}x &= \int e^{-x^2}8x\,\mathrm{d}x \\ y \cdot e^{-x^2} &= \int e^{-x^2}8x\,\mathrm{d}x\end{aligned}$$

Integrate completely to find $y$

$$y \cdot e^{-x^2} = \int e^{-x^2} 8x \, \mathrm{d}x = 8 \int e^{-x^2} x \, \mathrm{d}x$$

$$\left\{ \begin{array}{c} u = -x^2 \\ du = -2x dx \end{array} \right\} = 8 \int \frac{e^u x}{-2x} du$$

$$= -4 \int e^u du$$

$$e^{x^2} \left( y \cdot e^{-x^2} \right) = e^{x^2} \left( -4e^{-x^2} + C \right)$$

(b) $\boxed{y = Ce^{x^2} - 4}$

## 1(c) $\dfrac{du}{dv} = 3u + 2v$

We'll find $u(v)$. In standard form

$$\frac{du}{dv} - 3u = 2v$$
$$P(v) = -3, \ f(v) = 2v$$

### Short version

$$u \cdot e^{\int P(v)\,dv} = \int e^{\int P(v)\,dv} f(v)\,dv$$
$$e^{\int P(v)\,dv} = e^{-\int 3\,dv} = e^{-3v}$$
$$y \cdot e^{-3v} = \int e^{-3v} 2v\,dv$$

### Long version

The integrating factor (called $\phi$ to distinguish it from $u$)

$$\phi(x) = e^{\int P(v)\,dv} = e^{\int -3\,dv}$$
$$= e^{-3v}$$

multiplied through DE

$$\frac{du}{dv}e^{-3v} - 3e^{-3v}u = e^{-3v}2v$$

Differentiate and substitute

$$\frac{d\phi}{dv} = -3e^{-3v}$$
$$\frac{du}{dv}e^{-3v} + \frac{d\phi}{dv}u = e^{-3v}2v$$

which is the derivative

$$\frac{d}{dv}\left[u \cdot e^{-3v}\right] = e^{-3v}2v$$

then

$$\int \frac{d}{dv}\left[u \cdot e^{-3v}\right] dv = \int e^{-3v} 2v\,dv$$
$$u \cdot e^{-3v} = 2\int e^{-3v} v\,dv$$
$$\left\{ \begin{array}{ll} p = v & dq = e^{-3v}\,dv \\ dp = dv & q = -\frac{1}{3}e^{-3v} \end{array} \right\} = 2\left(-\frac{1}{3}ve^{-3v} - \int -\frac{1}{3}e^{-3v}\,dv\right)$$
$$= -\frac{2}{3}ve^{-3v} + \frac{2}{3}\int e^{-3v}\,dv$$
$$u \cdot e^{-3v} = -\frac{2}{3}ve^{-3v} - \frac{2}{9}e^{-3v} + C$$
$$e^{3v}\left(u \cdot e^{-3v}\right) = e^{3v}\left(C - \frac{2}{3}ve^{-3v} - \frac{2}{9}e^{-3v}\right)$$

$$\text{(c)} \quad \boxed{u = Ce^{3v} - \frac{2}{3}v - \frac{2}{9}}$$

## 2. $\dfrac{dv}{dt} = 9.8 - 2v, \quad v(0) = -5$

This ODE models an object's acceleration in one dimension. It comprises a constant gravitational acceleration in the positive $v$ direction, implying that the positive direction is down, and a drag term opposing the direction of velocity $v$. The object's mass is given as 1 kg, but this is irrelevant to us because it's already baked into the coefficient on the drag term ($k/m = 2$).

The object has an initial velocity at $t = 0$ of $-5\,\mathrm{m/s}$, implying that it's moving straight up. The net acceleration then is $9.8 - 2(-5) = 9.8 + 10\,\mathrm{m/s^2}$, meaning the object is initially slowing at more than twice the rate due to gravity.

**(a)** We're asked to find the time that the object stops rising due to having been thrown and starts falling due to gravity. This point is when the height curve reaches its local minimum (up is negative), or when the velocity $v = 0$. With a single variable model, where acceleration is a function of only time, we would just integrate acceleration to get a velocity function; but in this case we're trying to integrate $v(t)$ to find its definition:

$$v(t) = \int \frac{dv}{dt}\,\mathrm{d}t = \int 9.8 - 2v(t)\,\mathrm{d}t = 9.8t - 2\int v(t)\,\mathrm{d}t$$

We have a separable linear differential equation so we can find $v(t)$ by either of two methods:

### Variable separation

$$\frac{\cancel{dt}}{9.8 - 2v}\left(\frac{dv}{\cancel{dt}}\right) = \cancel{(9.8 - 2v)}\frac{dt}{\cancel{9.8 - 2v}}$$

$$\int \frac{1}{9.8 - 2v}\,\mathrm{d}v = \int 1\,\mathrm{d}t \qquad \begin{aligned} u &= 9.8 - 2v \\ du &= -2\,dv \end{aligned}$$

$$\begin{aligned} -\frac{1}{2}\int \frac{1}{u}\,\mathrm{d}u &= t + C \\ \ln|9.8 - 2v| &= -2t - 2C \\ 9.8 - 2v &= e^{-2t-2C} \\ v &= -\frac{1}{2}\left(e^{-2t-2C} - 9.8\right) \\ v(t) &= 4.9 - \frac{1}{2}e^{-2t-2C} \end{aligned} \tag{10}$$

### Integrating factor

$\frac{dv}{dt} = 9.8 - 2v$ in standard linear form is

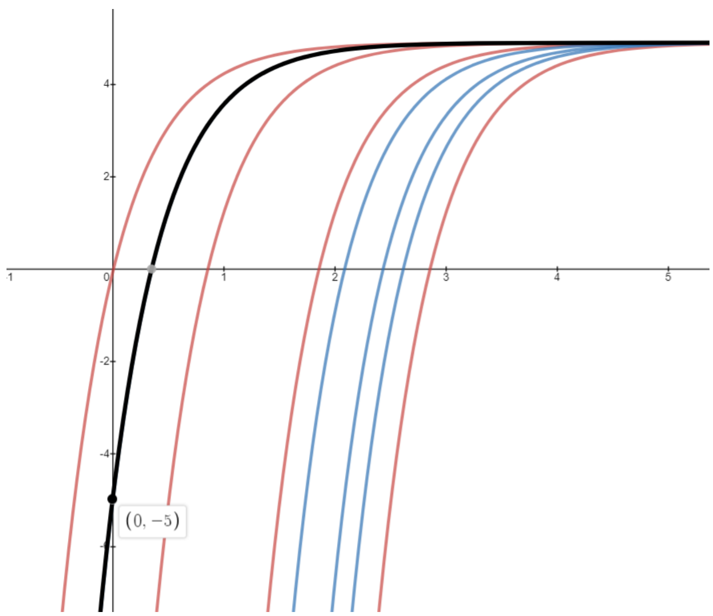
$$\frac{dv}{dt} + 2v = 9.8 \qquad P(t) = 2,\ f(t) = 9.8$$

so

$$\begin{aligned} v \cdot e^{\int P(t)\,\mathrm{d}t} &= \int e^{\int P(t)\,\mathrm{d}t} f(t)\,\mathrm{d}t \\ v \cdot e^{2t} &= \int e^{2t} \cdot 9.8\,\mathrm{d}t = 4.9e^{2t} + C \\ v(t) &= 4.9 + Ce^{-2t} \end{aligned} \tag{11}$$

Each of these solutions for $v(t)$ describes a family of functions all of which change in the same way with $t$. Although those functions have similar behavior from one point to the next, the points that they pass through differ with $C$. We need the specific functions in those families that pass through the point $v(0) = -5$, so we find whatever $C$ satisfies that initial condition. Here we use the function from (11), but the process is identical for (10):

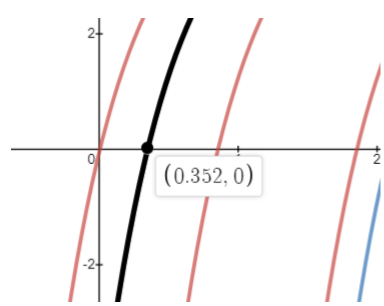
$$\begin{aligned} v(0) &= 4.9 + C \cdot e^{-2(0)} \\ -5 &= 4.9 + C \cdot e^{0} \\ -9.9 &= C \\ &\text{so} \\ v(t) &= 4.9 - 9.9e^{-2t} \end{aligned} \tag{12}$$




Functions from the families described by (10, red) and (11, blue). The function (12, black) satisfies the initial condition $v(0) = -5$.

The turnaround point, when the object stops rising and begins falling, occurs when $v(t) = 0$. Solving (12) for $t$ gives us the turnaround time $t_1$

$$\begin{aligned} v(t_1) = 0 &= 4.9 - 9.9e^{-2t_1} \\ e^{-2t_1} &= \frac{4.9}{9.9} \\ t_1 &= -\frac{1}{2}\ln 0.\overline{49} \\ t_1 &\approx \boxed{0.3516\,\mathrm{s}} \end{aligned}$$



**(b)** We're also asked how high the object goes before turning around. For this we just integrate its velocity from the initial time to the turnaround point:

$$\begin{aligned}\int_0^{t_1} v(t)\,\mathrm{d}t &= \int_0^{t_1} 4.9 - 9.9e^{-2t}\,\mathrm{d}t \\ &= \left[4.9t + 4.95e^{-2t} + C\right]_0^{t_1} \qquad (13)\end{aligned}$$

The integration constant $C$ in this case represents a constant displacement of the height gained after throwing; in other words $C$ sets the initial height. Because the initial height displaces both the starting and ending height, we see that it drops out of the height gained:

$$\begin{aligned}h_1 &= \left[4.9t + 4.95e^{-2t} + C\right]_0^{t_1} \\ &= \left(4.9t_1 + 4.95e^{-2t_1} + C\right) - \left(4.9\cdot 0 + 4.95e^{-2\cdot 0} + C\right) \\ &= 4.9t_1 + 4.95e^{-2t_1} - 4.95 + \cancel{C - C} \\ h_1 &\approx \boxed{-0.7769\,\mathrm{m}} \text{ (negative is up)}\end{aligned}$$

We assumed the units of the answers, but we should do the dimensional analysis to prove that the units make sense and to help us connect the equations to the physical model.

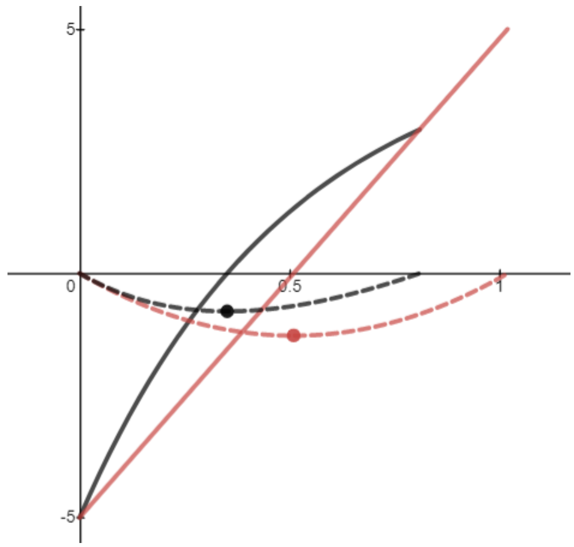
$$\frac{dv}{dt}\left[\frac{\mathrm{m}}{\mathrm{s}^2}\right] + 2\left[\frac{1}{\mathrm{s}}\right]\cdot v\left[\frac{\mathrm{m}}{\mathrm{s}}\right] = 9.8\left[\frac{\mathrm{m}}{\mathrm{s}^2}\right]$$

$$v\left[\frac{\mathrm{m}}{\mathrm{s}}\right]\cdot e^{\int P(t)\left[\cancel{\mathrm{s}^{-1}}\right]\mathrm{d}t\left[\cancel{\mathrm{s}}\right]} = \int e^{\int P(t)\left[\cancel{\mathrm{s}^{-1}}\right]\mathrm{d}t\left[\cancel{\mathrm{s}}\right]}\cdot f(t)\left[\frac{\mathrm{m}}{\mathrm{s}^{\cancel{2}}}\right]\mathrm{d}t\left[\cancel{\mathrm{s}}\right]$$

$$\int_0^{t_1} v(t)\left[\frac{\mathrm{m}}{\cancel{\mathrm{s}}}\right]\mathrm{d}t\left[\cancel{\mathrm{s}}\right] = \left[4.9\left[\frac{\mathrm{m}}{\cancel{\mathrm{s}}}\right]t\left[\cancel{\mathrm{s}}\right] + 4.95\left[\frac{\mathrm{m}}{\cancel{\mathrm{s}}}\right]e^{-2t}\left[\cancel{\mathrm{s}}\right] + C\left[\mathrm{m}\right]\right]_0^{t_1} = h_1\left[\mathrm{m}\right]$$

Taking our indefinite integral $h(t)$ from (13) we can plot the height of the object over time. Unlike before, the initial height doesn't cancel here–so we arbitrarily decide that the initial height is $h(0) = 0\,\text{m}$ and find $C$ to satisfy this:

$$\begin{aligned} h(t) &= 4.9t + 4.95e^{-2t} + C \\ h(0) = 0 &= 4.9 \cdot 0 + 4.95e^{0} + C \\ 0 &= 4.95 + C \\ C &= -4.95\,\text{m} \\ &\text{so} \\ h(t) &= 4.9t + 4.95e^{-2t} - 4.95 \end{aligned}$$




The velocity of our object with drag (solid black) and its height (dashed black) over time. The red curves show the velocity (solid red) and height (dashed red) of an object that doesn't experience drag.

The black velocity curve shows that drag slows the object considerably when its velocity is very negative, causing the black height curve to peak earlier than the red one. The red curve consequently reaches a higher maximum than the black curve does. Finally the object without drag returns to 0 m later because it fell from a higher height.

The object with drag isn't just slowed in its ascent; it's also slowed in its descent. We see this where the solid black velocity curve stops moving away from the red velocity line and approaches it again. In fact the object without drag will continue speeding up forever while the object with drag is sloping toward its terminal velocity.

**Extra stuff**

We can find the time when the object finishes its trip if we find the second point where $h(t) = 0$. This involves solving a polynomial with an exponential term, which I don't know how to do:

$$\begin{aligned} h(t) &= 4.9t + 4.95e^{-2t} - 4.95 = 0 \\ 4.95 &= 4.9t + 4.95e^{-2t} \end{aligned}$$

I considered finding a Taylor series for this function, hoping that we can find roots for even a degree four polynomial. The $n$th derivative of $h$ is easy to find at least.

$$\begin{aligned} h^0(t) &= 4.95e^{-2t} + 4.9t - 4.95 & h^0(0) &= 0 \\ h^1(t) &= -9.9e^{-2t} + 4.9 & h^1(0) &= -5 \\ h^2(t) &= 19.8e^{-2t} & h^2(0) &= 19.8 \\ h^3(t) &= -39.6e^{-2t} & h^3(0) &= -39.6 \\ h^4(t) &= 79.2e^{-2t} & h^4(0) &= 79.2 \end{aligned}$$

$$h(t) = \sum_{n=0}^{4} \frac{h^{(n)}(0)}{n!} t^n = -5t + \frac{19.8}{2}t^2 - \frac{39.6}{6}t^3 + \frac{79.2}{24}t^4$$

Even with four terms the approximation is not good enough to use, and a fifth degree polynomial is not obviously solvable, so I gave up on this. If you know a way to make it happen I'd love to see it. Anyway we already have the function in a numerical approximator so we can just use that.

The object is in the air for about 0.8105 seconds. Our Taylor approximation gives 0.7312 seconds so we're short by about 10%.