Quiz 2.1

1. $y'' - 8y' + 12y = 0$

Substituting the differential operator $D^n y = \dfrac{d^n y}{dx^n} = y^{(n)}$ and factoring, we have

$$D^2 y - 8Dy + 12y = \left(D^2 - 8D + 12\right) y = p(D)y = 0$$

where $p(D)$ is a polynomial differential operator. With roots of the polynomial $p(D)$ we can find the fundamental set of solutions:

$$p(D) = D^2 - 8D + 12 = (D-2)(D-6) = 0$$

$$D = 2, \quad D = 6$$

so our solution is the linear combination

$$\boxed{y = C_1 e^{2x} + C_2 e^{6x}}.$$

Why the fundamental solutions have the form Ae^{ax} is not super obvious to me. As long as a is distinct then it meets the linear independence requirement; but if a root has multiplicity greater than one then a is not distinct and the terms aren't linearly independent. This is fixed by using $C_n x^{n-1} e^{ax}$, but I don't really understand where that comes from. I guess it's a series.

I can at least prove that Ce^{ax} is a solution to the degree-one ODE $y' - ay = 0$. If we let $z = e^{-ax} y$ then

$$\begin{aligned} z' &= -aye^{-ax} + y'e^{-ax} \\ &= e^{-ax}\left(y' - ay\right). \end{aligned}$$

Now $y' - ay = 0$, so $y' = ay$ and $z' = e^{-ax}(ay - ay) = 0$. Finally

$$z = \int z' \, \mathrm{d}x = \int 0 \, \mathrm{d}x = 0 + C = C,$$

and z is defined as $e^{-ax}y$, so $z = C = e^{-ax}y$ and $y = Ce^{ax}$. This presupposed that we chose z as an exponential, so I still don't see where that choice comes from.

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2. $y'' - 9y = 0$

Again apply the differential operator and factor the perfect squares:

$$\left(D^2 - 9\right) y = \left(D^2 - 3^2\right) y = (D-3)(D+3)\, y = 0$$

giving us the roots $D = \pm 3$ and the solution

$$\boxed{y = C_1 e^{3x} + C_2 e^{-3x}}.$$

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3. $y'' + 9y = 0$

This time $p(D)$ doesn't have real factors:

$$\begin{aligned} \left(D^2+9\right) y &= 0 \\ D^2 &= -9 \\ |D| &= \sqrt{-9} = \sqrt{9}\sqrt{-1} \\ D &= \pm 3i. \end{aligned}$$

This gives us the complex exponential solution

$$y = C_1 e^{3ix} + C_2 e^{-3ix}$$

which simplifies by Euler's formula to the real general solution

$$y = C_1 e^{0x}\cos 3x + C_2 e^{0x}\sin 3x$$

$$\boxed{y = C_1 \cos 3x + C_2 \sin 3x}.$$

Euler's formula

$$e^{ix} = \cos x + \sin x \text{ and } e^{-ix} = \cos x - \sin x$$

We found the general solution as a linear combination of two linearly independent real solutions y_1 and y_2. To find these we first consider the complex solutions u and v, where

$$\begin{aligned} u = e^{(a+bi)x} &= e^{ax}e^{bix} \\ &= e^{ax}\left(\cos bx + i\sin bx\right) \end{aligned}$$

and

$$\begin{aligned} v = e^{(a-bi)x} &= e^{ax}e^{-bix} \\ &= e^{ax}\left[\cos\left(-bx\right) + i\sin\left(-bx\right)\right] \\ &= e^{ax}\left(\cos bx - i\sin bx\right). \end{aligned}$$

We want real solutions, which we find by first adding

$$\begin{aligned} u + v &= e^{ax}\left[\left(\cos bx + i\sin bx\right) + \left(\cos bx - i\sin bx\right)\right] \\ &= e^{ax}\left(2\cos bx\right) \end{aligned}$$

then subtracting

$$\begin{aligned} u - v &= e^{ax}\left[\left(\cos bx + i\sin bx\right) - \left(\cos bx - i\sin bx\right)\right] \\ &= e^{ax}\left(2i\sin bx\right) \end{aligned}$$

so that we have y_1 and y_2 as linear combinations of these

$$\begin{aligned} y_1 &= u + v = e^{ax}\left(2\cos bx\right) \\ y_2 &= u - v = e^{ax}\left(2i\sin bx\right), \end{aligned}$$

which we make real (and simplify) by moving complex factors into the combination weights

$$\begin{aligned} y_1 &= \frac{1}{2}u + \frac{1}{2}v = e^{ax}\frac{(\cancel{2}\cos bx)}{\cancel{2}} = e^{ax}\cos bx \\ y_2 &= \frac{1}{2i}u - \frac{1}{2i}v = e^{ax}\frac{(\cancel{2i}\sin bx)}{\cancel{2i}} = e^{ax}\sin bx \end{aligned}$$

Finally we have our real general solution as the combination of these

$$y = C_1 y_1 + C_2 y_2 = e^{ax}\left(C_1\cos bx + C_2\sin bx\right).$$

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4. $y'' - 8y' + 20y = 0$

This ODE also has no real roots for $p(D)$, so we use the quadratic formula with $a = 1, b = -8, c = 20$:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{8 \pm \sqrt{64 - 4 \cdot 1 \cdot 20}}{2} = \frac{8 \pm \sqrt{-16}}{2} = 4 \pm 2i$$

yielding the complex solution

$$y = K_1 e^{(4+2i)x} + K_2 e^{(4-2i)x}.$$

Using the method from question $\boxed{3}$ gives us the real solution

$$\begin{aligned} y_1 &= e^{ax} \cos bx = e^{4x} \cos 2x \\ y_2 &= e^{ax} \sin bx = e^{4x} \sin 2x. \end{aligned}$$

$$y = C_1 y_1 + C_2 y_2 = \boxed{e^{4x} \left(C_1 \cos 2x + C_2 \sin 2x\right)}.$$

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5. $4y'' - 13y' + 3y = 0$

Again using the quadratic formula with $a = 4, b = -13, c = 3$,

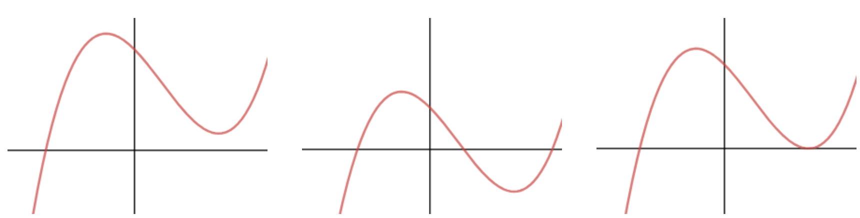
$$\frac{13 \pm \sqrt{13^2 - 4 \cdot 4 \cdot 3}}{8} = \frac{13 \pm 11}{8} = \frac{13}{8} \pm \frac{11}{8}$$

the roots are 3 and $1/4$ so the factors of $p(D)$ are $(4D - 1)$ and $(D - 3)$ (though we don't need them). The solution is

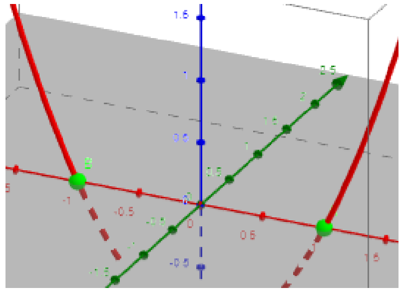
$$\boxed{y = C_1 e^{\frac{1}{4}x} + C_2 e^{3x}}.$$

6. $y''' - 2y'' = 0$

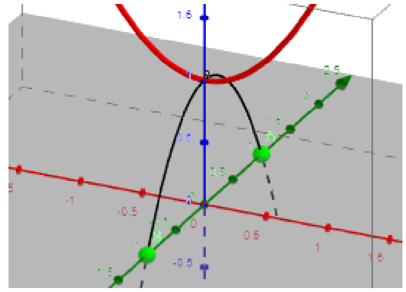
This is the first degree three $p(D)$ we'll deal with. A degree n polynomial has exactly n roots, counting multiplicities, so we expect our solution to involve three terms, real or complex. With a cubic function we can find either one real and two complex roots or three real roots (either all roots are real, or there is an even number of complex roots; each complex root comes in a conjugate pair).



One real and two complex roots | Three real roots | Three real roots, one with multiplicity 2



This parabola has roots on the x-axis corresponding to the real factors $(x+1)(x-1) = x^2 - 1$.



This parabola intersects the complex plane at i and $-i$ on the green (imaginary) axis. These roots correspond to the complex factors $(x+i)(x-i) = x^2 + 1$.
All values on the red curve are real on the xy-plane, but all of its factors are complex.

This polynomial factors easily:

$$\left(D^3 - 2D^2\right) y = \left[D \cdot D\left(D-2\right)\right] y = 0$$

yielding the real roots $D = 0$ with multiplicity 2 and $D = 2$. If a root r has multiplicity n then the solution terms of the fundamental set for that root are

$$\sum_{i=0}^{n-1} C_{i+1} x^i e^{rx} = C_1 e^{rx} + C_2 x e^{rx} + \cdots + C_{n-1} x^{n-1} e^{rx}.$$

In our case we have two distinct roots, $(r, n) = (0, 2)$ and $(2, 1)$; our solution then is

$$y = \left(C_1 x^0 e^{0x} + C_2 x^1 e^{0x}\right) + \left(C_3 x^0 e^{2x}\right) = \boxed{C_1 + C_2 x + C_3 e^{2x}}.$$

7. $8y''' - 12y'' - 18y' + 27y = 0$

The two pairs of terms here share a common ratio: $8/-12 = -18/27 = -2/3$. Factor by grouping (be mindful of how the signs distribute):

$$\begin{aligned} \left(8D^3 - 12D^2\right) y - \left(18D - 27\right) y &= 0 \\ 4D^2\left(2D-3\right) y - 9\left(2D-3\right) y &= 0 \\ \left(4D^2 - 9\right)\left(2D-3\right) y &= 0 \end{aligned}$$

Notice that $4D^2 - 9$ is a difference of squares $(2D)^2$ and 3^2; we can continue factoring as

$$\left(2D+3\right)\left(2D-3\right)\left(2D-3\right) y = \left(2D+3\right)\left(2D-3\right)^2 y = 0.$$

Our roots are $D = \frac{3}{2}$ with multiplicity 2 and $D = -\frac{3}{2}$ and the solution is

$$\boxed{y = e^{\frac{3}{2}x}\left(C_1 + C_2 x\right) + C_3 e^{-\frac{3}{2}x}}.$$

8. $y''' + 6y'' + 13y' + 10y = 0$

The *Rational Root Theorem* says that

> if a degree-n polynomial with integer coefficients has any rational roots they must be of the form $\pm\frac{a_0}{a_n}$, where a_0 is a factor of the coefficient on the lowest-degree term and a_n on the highest.

For $p(D) = D^3 + 6D^2 + 13D + 10$, $a_0 = 10$ and $a_n = 1$; we know then that it must have roots of the form $\pm a$ where a is a factor of 10 and it's divided by 1. Factors of 10 are $\{1, 2, 5, 10\}$, so we test if these are roots of the polynomial. All of the terms are positive, so we know any positive value is not going to sum to zero and we can test exclusively negative values.

$$\begin{aligned}
(-1)^3 + 6(-1)^2 + 13(-1) + 10 &= -1 + 6 - 13 + 10 &&= 2 \neq 0 \\
(-2)^3 + 6(-2)^2 + 13(-2) + 10 &= -8 + 24 - 26 + 10 &&= 0
\end{aligned}$$

We can simplify the polynomial by factoring this root out of it using polynomial long division. Synthetic division offers a shorthand to simplify the process.

Long

$$\begin{array}{r}
D^2 \;+4D\;+5 \\
D+2\,\overline{)\;D^3+6D^2+13D+10} \\
-\,D^3-2D^2 \qquad\qquad \\
\hline
4D^2+13D \qquad \\
-\,4D^2\;-8D \qquad \\
\hline
5D+10 \\
-\,5D-10 \\
\hline
0
\end{array}$$

Synthetic

$$\begin{array}{r|rrrr}
 & 1 & 6 & 13 & 10 \\
-2 & & -2 & -8 & -10 \\
\hline
 & 1 & 4 & 5 & 0
\end{array}$$

$1D^2 + 4D + 5$

As long as the polynomial divides evenly then we can factor it in this way.

$$D^3 + 6D^2 + 13D + 10 = (D+2)\left(D^2 + 4D + 5\right) = 0$$

We factor the remaining trinomial with the quadratic formula,

$$\frac{-4 \pm \sqrt{16 - 4 \cdot 1 \cdot 5}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = -2 \pm i,$$

giving us the real root $D = -2$ and a pair of conjugate complex roots $D = -2 + i$ and $D = -2 - i$. Following the scheme from question $\boxed{3}$ our solution is

$$\boxed{y = C_1 e^{-2x} + e^{-2x}\left(C_2 \cos x + C_3 \sin x\right)}.$$